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Super-simple Steiner pentagon systems☆

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Abstract

A Steiner pentagon system of order v (SPS(v)) is said to be super-simple if its underlying $(v, 5, 2)$-BIBD is super-simple; that is, any two blocks of the BIBD intersect in at most two points. In this paper, it is shown that the necessary condition for the existence of a super-simple SPS(v); namely, $v \geqslant 5$ and $v \equiv 1$ or $5 \pmod{10}$ is sufficient, except for $v = 5$, 15 and possibly for $v = 25$. In the process, we also improve an earlier result for the spectrum of super-simple $(v, 5, 2)$-BIBDs, removing all the possible exceptions. We also give some new examples of Steiner pentagon packing and covering designs (SPPDs and SPCDs).

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1. Introduction

Let K_n be the complete undirected graph with n vertices. A *pentagon system* (PS) of order n is a pair $(K_n, \mathscr{B})$, where $\mathscr{B}$ is a collection of edge disjoint pentagons which partition the edges of K_n. A *Steiner pentagon system* (SPS) of order n is a pentagon system $(K_n, \mathscr{B})$ with the additional property that every pair of vertices is joined by both one path of length 1 in exactly one pentagon of $\mathscr{B}$, and also one path of length 2 in exactly one pentagon of $\mathscr{B}$. A (v, k, λ) *balanced incomplete block design* (briefly (v, k, λ)-BIBD) is a pair $(X, \mathscr{B})$, where X is a v-set of *points* and $\mathscr{B}$ is a collection of k-element subsets of X, called *blocks*, such that every pair of points of X is contained in exactly λ blocks of $\mathscr{B}$. It is fairly evident that any SPS of order v will give a $(v, 5, 2)$-BIBD. However, the converse is not necessarily true. It is known [13] that the spectrum of SPSs is precisely the set of all $v \equiv 1$ or $5 \pmod{10}$, except $v = 15$ for which there exists no such system or $(v, 5, 2)$-BIBD. SPSs are also known to be useful in the construction of other types of designs such as perfect Mendelsohn designs with block-size five (see [7]) and authentication perpendicular arrays (see [12]).

A (v, k, λ)-BIBD is called *super-simple* if any two blocks of the BIBD intersect in at most two points. A Steiner pentagon system of order v (SPS(v)) is said to be super-simple if its underlying $(v, 5, 2)$-BIBD is super-simple. The

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concept of super-simple BIBDs was introduced by Gronau and Mullin in [11]. This concept was further studied in subsequent papers, including [10,8], where the spectrum of super-simple $(v, 5, 2)$-BIBDs was investigated with the following results:

Theorem 1.1. *There exists a super-simple* $(v, 5, 2)$*-BIBD if and only if* $v \equiv 1$ *or* $5 \pmod{10}$, *except for* $v \in \{5, 15\}$, *and possibly for* $v \in \{115, 135, 195, 215, 231, 285, 365, 515\}$.

In this paper, it is shown that the necessary condition for the existence of a super-simple SPS(v), namely, $v \geqslant 5$ and $v \equiv 1$ or $5 \pmod{10}$ is sufficient, except for $v = 5$ and 15 and possibly for $v = 25$. In the process, we consequently remove all the possible exceptions for super-simple $(v, 5, 2)$-BIBDs in Theorem 1.1. Apart from being of interest in their own right, super-simple designs have appeared as suborthogonal double covers of certain types of graphs (see, for example, [9]).

2. Auxiliary designs

A *pairwise balanced design* (or PBD) is a pair $(X, \mathscr{A})$ such that X is a set of elements called *points*, and $\mathscr{A}$ is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every pair of points is in a unique block of $\mathscr{A}$. Often PBDs are called linear spaces. If v is a positive integer and K is a set of positive integers, each of which is greater than or equal to two, then we say that $(X, \mathscr{A})$ is a $(v, K) - PBD$ if $| X | = v$, and $| A | \in K$ for every $A \in \mathscr{A}$. When $K = \{k\}$, we normally write k instead of K. Also, we denote $B(K) = \{v$: *there exists a* $(v, K) - PBD\}$.

Let λ be a positive integer. A (K, λ) *group divisible design* (or GDD) is a triple $(X, \mathscr{G}, \mathscr{A})$, which satisfies four properties: (1) $\mathscr{G}$ is a partition of X into subsets called *groups*; (2) $\mathscr{A}$ is a set of subsets of X (called *blocks*) with sizes from K; (3) no two points in the same group appear in any block; (4) any two of points from distinct groups occur together in λ blocks. The parameter λ can be omitted if it equals 1.

The *group type* of a (K, λ)-GDD$(X, \mathscr{G}, \mathscr{A})$ is a multiset $\{| G |: G \in \mathscr{G}\}$. We use exponential notation to describe group types: a group type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ denotes u_i occurrences of a group of size g_i for $1 \leqslant i \leqslant s$. Notationally, we permit $g_i = g_j$ when $i \neq j$. Groups of size 0 can be added as convenient. If $K = \{k\}$, we write k instead of K. A *transversal design*, $TD_\lambda(k, n)$, is a $(k, \lambda) - GDD$ of type n^k; the parameter λ is usually omitted if it equals 1. It is well known that a $TD(k, n)$ is equivalent to $k - 2$ mutually orthogonal Latin squares (*MOLS*) of order n. For a list of lower bounds on the number of *MOLS* of all orders up to 10 000, see [3].

Let K be a set of positive integers, and let k be a positive integer. The notation $(v, K \cup \{k^\star\})$-PBD denotes a PBD with a distinguished block of size k and other block sizes in K. Only if $k \in K$, can there be more than one block of size k.

Let S be a set and $\mathscr{H} = \{S_1, S_2, \ldots, S_n\}$ be a set of subsets of S. A *holey Steiner pentagon system* having *hole set* $\mathscr{H}$ is a triple $(S, \mathscr{H}, \mathscr{B})$ where $\mathscr{B}$ is a collection of pentagons, satisfying the following properties:

(1) Two vertices from the same hole S_i do not occur together in any pentagon of $\mathscr{B}$.
(2) Two vertices from different holes S_i and S_j $(i \neq j)$ are joined by a path of length 1 in exactly one pentagon of $\mathscr{B}$, and also by a path of length 2 in exactly one pentagon of $\mathscr{B}$.

The *order* of the system is $|S|$.

If $\mathscr{H} = \{S_1, S_2, \ldots, S_n\}$ is a partition of S, then we simply denote the system by HSPS(T), where T is the *type* and defined to be the multiset $\{|S_i| : 1 \leqslant i \leqslant n\}$. The sets S_i are called *holes*. We shall use an "exponential" notation to describe types: so type $t_1^{u_1} \ldots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leqslant i \leqslant k$, in the multiset. It is clear that an SPS of order v (or SPS(v)) is equivalent to an HSPS(1^v). Also an HSPS of type $(1^{v-k} k^1)$ is called an *incomplete Steiner pentagon system* and denoted by ISPS(v, k).

The existence of HSPSs of uniform type h^n has been investigated by Abel et al. in [1]. The following known results are given in [1].

Theorem 2.1 (*Abel et al. [1]*). *Necessary conditions for an HSPS of type* h^n *to exist are*: (1) $n \geqslant 5$ (2) $n(n-1)h^2 \equiv 0 \pmod 5$ *and* (3) *if h is odd, then n is odd. These conditions are also sufficient, except possibly in the following* *cases*:

(1) *when* $(h, n) \in \{(6, 6), (6, 36), (30, 18), (30, 22), (30, 24)\}$;
(2) *when* $n = 15$, *and* $h = 9$ *or* $h \in \{t : t \equiv 1$ *or* $5 \pmod 6, t \not\equiv 0 \pmod 5\}$;
(3) *when* $(h, n) \in \{(15, 19), (15, 23), (15, 27)\}$.

It is well known that the existence of an HSPS of type T implies the existence of a (5, 2)-GDD of type T. We shall call an HSPS of type T super-simple if its underlying (5, 2)-GDD of type T is super-simple; that is, any two blocks of the GDD intersect in at most two points.

3. Direct constructions

The constructions used in this paper will combine both direct and recursive methods. For our direct constructions, we shall adopt the standard approach of using finite abelian groups to generate the set of pentagons for any given HSPS. That is, instead of listing all of the pentagons, we give the vertex set X, plus a set of base (or starter) pentagons, and generate the other pentagons by an additive group G. For $g = |G|$, the notation $+t \pmod g$ after the base pentagons means the base pentagons should be developed by adding multiples of $t \pmod g$ to them; we omit t if it equals 1. Occasionally, a few base pentagons are given as 'short', that is, their g translates are not all distinct, and should be included once only. Also, sometimes an HSPS will contain one or more infinite points; unless otherwise stated, these should remain unaltered when developing a base pentagon over G. Finally, in most cases, holes of the HSPS will be translates of an obvious additive subgroup of G, or will consist of the infinite points; when this is the case, we do not specify them.

Lemma 3.1. *There exist super-simple HSPSs of types* 3^5, 5^5 *and* 5^7.

Proof. For type 3^5, let $X = Z_{15}$, and develop the following pentagons (mod15). The second pentagon is 'short', and generates only 3 pentagons.

$$(0, 1, 13, 9, 2), \quad (0, 6, 12, 3, 9).$$

For type 5^5, let $X = Z_5 \times Z_5$, and develop the following two pentagons (mod (5, 5)):

$$((0, 0), (1, 1), (2, 4), (3, 4), (4, 1)), \quad ((0, 0), (2, 2), (4, 3), (1, 3), (3, 2)).$$

For type 5^7, let $X = Z_{35}$, and develop the following three pentagons (mod 35):

$$(0, 31, 20, 19, 29), \quad (0, 3, 1, 13, 30), \quad (0, 9, 24, 5, 27). \quad \square$$

Lemma 3.2. *There exist super-simple HSPSs of types* 2^n *for* $n = 6, 10, 11, 15, 16, 20, 21$.

Proof. For $n = 6$, the HSPS given in [1] is super-simple. For $n = 11, 16, 21$, let $X = Z_{2n}$ and develop the following base pentagons:

$n = 11$:	(0, 3, 1, 15, 19),	(0, 6, 14, 4, 5),	(0, 7, 12, 5, 21),	(0, 4, 17, 7, 20),	+2 (mod 22).
$n = 16$:	(0, 11, 19, 28, 29),	(0, 30, 11, 1, 5),	(0, 7, 27, 9, 15),		(mod 32).
$n = 21$:	(0, 9, 5, 23, 11),	(0, 2, 12, 20, 15),	(0, 1, 4, 17, 36),	(0, 14, 34, 41, 25),	(mod 42).

For $n = 10, 15, 20$, let $X = Z_{2n-2} \cup \{\infty_0, \infty_1\}$ and develop the following base pentagons. For $n = 15, 20$, replace ∞_0 by ∞_1 when adding odd values to the last base pentagon.

$n = 10$:	(0, 17, 1, 5, 15),	(0, 12, 2, 4, 5),	$(0, 7, 12, 15, \infty_0)$,	$(0, 14, 7, 13, \infty_1)$,	+2 (mod 18).
$n = 15$:	(0, 2, 5, 4, 13),	(0, 5, 1, 21, 11),	$(0, 6, 13, 25, \infty_0)$,		(mod 28).
$n = 20$:	(0, 2, 6, 12, 11),	(0, 8, 18, 30, 5),	(0, 18, 34, 11, 35),	$(0, 9, 2, 23, \infty_0)$,	(mod 38). $\square$

Lemma 3.3. *There exist super-simple HSPSs of type* 4^n *for* $n = 5, 6, 10, 11, 16, 20$.

Proof. For $n = 6$, 11 and 16, Let $X = Z_{4n}$ and develop the following base pentagons.

$n = 6$:	(0, 5, 20, 21, 7),	(0, 3, 19, 17, 4),			(mod 24).
$n = 11$:	(0, 9, 5, 23, 15),	(0, 2, 8, 18, 43),	(0, 5, 42, 29, 17),	(0, 16, 40, 17, 3),	(mod 44).
$n = 16$:	(0, 1, 3, 40, 47),	(0, 4, 14, 26, 35),	(0, 5, 11, 41, 56),	(0, 24, 63, 17, 61),	
	(0, 38, 60, 29, 50),	(0, 23, 59, 8, 53),			(mod 64).

For $n = 5$, 10, 20, let $X = Z_{4n-4}$ and develop the following base pentagons. For $n = 20$ and $x = 1, 2, 3$, replace ∞_0 by ∞_x when adding any value $\equiv x \pmod 4$ to a base pentagon.

$n = 5$:	$(0, 1, 2, 7, \infty_0)$,	$(0, 2, 5, 11, \infty_1)$,	$(0, 13, 6, 15, \infty_2)$,	$(0, 6, 1, 3, \infty_3)$,	+2 (mod 16).
$n = 10$:	(0, 4, 26, 16, 11),	(0, 1, 5, 25, 15),	(0, 5, 13, 35, 33),	(0, 24, 32, 2, 25),	
	$(0, 7, 8, 11, \infty_0)$,	$(0, 2, 31, 19, \infty_1)$,	$(0, 13, 30, 15, \infty_2)$,	$(0, 16, 33, 3, \infty_3)$,	+2 (mod 36).
$n = 20$:	(0, 1, 26, 40, 67),	(0, 2, 7, 23, 70),	(0, 11, 28, 31, 43),	$(0, 18, 52, 22, \infty_0)$,	
	(0, 36, 75, 9, 50),	(0, 53, 74, 6, 69),	(0, 45, 65, 33, 48),	$(25, 1, 55, 59, \infty_0)$,	(mod 76). □

Lemma 3.4. *There exist super-simple HSPSs of type* 10^n *for* $n = 5, 7, 8$.

Proof. Take $X = Z_{10(n-1)} \cup \{\infty_0, \infty_1, \ldots, \infty_9\}$ and develop the following base pentagons. For $n = 7$, replace ∞_0 by ∞_y (for $1 \leqslant y \leqslant 5$) when adding any value $\equiv y \pmod 6$ to a base pentagon. Also, for $n = 8$ and x even, or $n = 7$ and $x = 6, 8$, replace ∞_x by ∞_{x+1} when adding odd values to a base pentagon.

$n = 5$:	$(0, 26, 5, 7, \infty_0)$,	$(0, 1, 2, 11, \infty_1)$,	$(0, 3, 6, 17, \infty_2)$,	$(0, 2, 9, 31, \infty_3)$,	
	$(0, 35, 14, 1, \infty_4)$,	$(0, 6, 23, 33, \infty_5)$,	$(0, 13, 30, 35, \infty_6)$,	$(0, 10, 39, 25, \infty_7)$,	
	$(0, 15, 22, 13, \infty_8)$,	$(0, 18, 3, 37, \infty_9)$,			+2 (mod 40).
$n = 7$:	(0, 28, 2, 1, 39),	(0, 4, 7, 14, 23),	$(0, 47, 22, 39, \infty_6)$,	$(0, 2, 51, 5, \infty_8)$,	
	$(0, 15, 35, 43, \infty_0)$,	$(2, 18, 49, 22, \infty_0)$,	$(3, 44, 34, 29, \infty_0)$,		(mod 60).
$n = 8$:	(0, 2, 34, 22, 66),	(0, 6, 40, 45, 30),	(0, 31, 41, 19, 8),	$(0, 43, 2, 59, \infty_0)$,	
	$(0, 46, 43, 61, \infty_2)$,	$(0, 47, 66, 65, \infty_4)$,	$(0, 54, 17, 67, \infty_6)$,	$(0, 61, 44, 69, \infty_8)$,	(mod 70). □

Lemma 3.5. *For* $u = 0, 4, 8, 10$, *there exists a super-simple HSPS of type* 8^5u^1.

Proof. For $u = 8$ (i.e. type 8^6), we can apply Construction 4.1, using a super-simple HSPS(2^6) from Lemma 3.2 and a TD(5, 4). For the others, let $X = Z_{40} \cup \{\infty_0, \infty_1, \ldots, \infty_{u-1}\}$, and develop the following base pentagons (mod 40). For $u = 0$ or 10, the last pentagon is short, and generates only 8 pentagons. For $u = 4$ and $x = 0, 2$, replace ∞_x by ∞_{x+1} when adding odd values to a base pentagon. For $u = 10$, $x = 0, 4$, and $y = 0, 1, 2, 3$, replace ∞_x by ∞_{x+y} when adding any value $\equiv y \pmod 4$ to a base pentagon. For $u = 10$, replace ∞_8 by ∞_9 when adding any value $\equiv 4, 5, 6, 7 \pmod 8$ to a base pentagon.

$u = 0$:	(0, 7, 31, 28, 29),	(0, 4, 23, 37, 31),	(0, 12, 34, 11, 38),	(0, 8, 16, 24, 32).	
$u = 4$:	(0, 9, 13, 11, 37),	(0, 16, 4, 22, 23),	$(0, 7, 26, 39, \infty_0)$,	$(0, 6, 17, 9, \infty_2)$.	
$u = 10$:	$(2, 1, 8, 24, \infty_0)$,	$(3, 15, 6, 17, \infty_0)$,	$(0, 19, 33, 11, \infty_4)$,	$(2, 8, 6, 29, \infty_4)$,	
	$(0, 13, 9, 12, \infty_8)$,	(0, 8, 16, 24, 32).			□

Lemma 3.6. *There exist super-simple HSPSs of type* $g^m u^1$ *for* $(g, m, u) = (1, 20, 3), (1, 30, 3), (1, 62, 7), (1, 70, 3), (1, 82, 7), (1, 102, 7), (2, 12, 4), (2, 15, 4), (2, 15, 6), (2, 17, 4), (2, 38, 8), (4, 8, 6), (4, 12, 8), (4, 13, 6), (4, 20, 2), (4, 23, 6), (4, 28, 6), (4, 33, 6), (4, 38, 6), (6, 5, 6), (6, 5, 8)$.

Proof. Take $X = Z_{gm} \cup \{\infty_0, \infty_1, \ldots, \infty_{u-1}\}$ and develop the following base pentagons. In most cases, for each base pentagon of the form (a, b, c, d, ∞_i) to be developed $\pmod{gm}$ (not +2 or +4 $\pmod{gm}$), ∞_i should be replaced by ∞_{i+1} when adding odd values to the pentagon if both $d - a$, $c - b$ are odd or when adding values $\equiv 2, 3 \pmod 4$ to the pentagon if both $d-a$, $c-b$ are equivalent to 2 (mod 4). For types $1^{30}3^1$, $2^{15}4^1$, 6^6, 6^58^1 and $z=0$, or $2^{15}6^1$ and $z=0,3$, replace ∞_z by ∞_{z+x} $(x=1,2)$ when adding any values $\equiv 2x$ (mod 6) to a base pentagon. For types $2^{15}4^1$ and 6^6, the first pentagon is short and generates only 3 pentagons.

$1^{20}3^1$:	(0, 15, 19, 11, 17),	(0, 10, 4, 8, 9),	(2, 13, 9, 17, 19),	(2, 8, 10, 6, 11),	
	(0, 5, 10, 13, ∞_0),	(2, 15, 8, 7, ∞_0),	(0, 12, 15, 9, ∞_1),	(2, 14, 13, 11, ∞_1),	
	(0, 13, 6, 7, ∞_2),	(2, 4, 15, 5, ∞_2),			+4 (mod 20).
$1^{30}3^1$:	(0, 26, 10, 2, 13),	(0, 7, 22, 27, 1),	(0, 6, 18, 21, 29),	(0, 17, 1, 13, 19),	
	(4, 14, 9, 18, ∞_0),	(15, 25, 23, 26, ∞_0),	(17, 24, 22, 1, ∞_0),		+2 (mod 30).
$1^{62}7^1$:	(0, 1, 11, 48, 26),	(0, 7, 1, 42, 23),	(0, 14, 46, 8, 50),	(1, 3, 25, 57, 16),	
	(0, 6, 50, 40, 29),	(0, 5, 55, 39, 57),	(1, 55, 27, 13, 36),	(0, 4, 32, 7, 3),	
	(0, 60, 31, 57, ∞_0),	(0, 13, 44, 43, ∞_1),	(0, 49, 52, 35, ∞_2),	(0, 54, 45, 21, ∞_3),	
	(0, 17, 60, 13, ∞_4),	(0, 35, 24, 17, ∞_5),	(0, 16, 25, 5, ∞_6),		+2 (mod 62).
$1^{70}3^1$:	(0, 1, 7, 54, 42),	(0, 25, 1, 52, 35),	(0, 60, 46, 48, 54),	(0, 63, 11, 67, 41),	
	(0, 22, 66, 20, 5),	(0, 9, 59, 63, 65),	(0, 36, 39, 61, 21),	(0, 37, 40, 10, 17),	
	(0, 39, 23, 15, 51),	(0, 52, 44, 3, 31),	(0, 49, 36, 51, 4),	(0, 50, 37, 69, 11),	
	(0, 33, 42, 41, ∞_0),	(0, 45, 56, 13, ∞_1),	(0, 32, 5, 65, ∞_2),		+2 (mod 70).
$1^{82}7^1$:	(0, 1, 21, 74, 46),	(0, 67, 1, 44, 79),	(0, 8, 46, 12, 62),	(0, 71, 73, 79, 37),	
	(0, 2, 42, 20, 13),	(0, 5, 27, 15, 25),	(0, 52, 19, 33, 51),	(0, 57, 48, 75, 72),	
	(0, 15, 65, 29, 59),	(0, 4, 68, 59, 11),	(0, 31, 35, 61, 7),	(0, 24, 10, 80, 41),	
	(0, 77, 32, 51, ∞_0),	(0, 6, 5, 49, ∞_1),	(0, 53, 30, 13, ∞_2),	(0, 55, 22, 39, ∞_3),	
	(0, 16, 77, 3, ∞_4),	(0, 26, 47, 71, ∞_5),	(0, 63, 76, 41, ∞_6),		+2 (mod 82).
$1^{102}7^1$:	(0, 1, 35, 86, 42),	(0, 65, 1, 84, 29),	(0, 11, 25, 68, 50),	(0, 49, 33, 44, 79),	
	(0, 2, 26, 94, 17),	(0, 3, 59, 87, 81),	(0, 4, 66, 28, 54),	(0, 99, 101, 9, 57),	
	(0, 6, 42, 20, 13),	(0, 5, 47, 43, 101),	(0, 56, 51, 69, 43),	(0, 53, 72, 55, 16),	
	(0, 71, 31, 81, 45),	(0, 12, 32, 23, 15),	(0, 7, 27, 57, 89),	(0, 8, 80, 6, 27),	
	(0, 9, 48, 79, ∞_0),	(0, 10, 97, 75, ∞_1),	(0, 23, 64, 39, ∞_2),	(0, 33, 62, 29, ∞_3),	
	(0, 32, 99, 21, ∞_4),	(0, 88, 41, 53, ∞_5),	(0, 37, 98, 71, ∞_6),		+2 (mod 102).
$2^{12}4^1$:	(0, 1, 19, 21, 11),	(0, 4, 9, 17, ∞_0),	(0, 7, 22, 1, ∞_2),		(mod 24).
$2^{15}4^1$:	(0, 6, 12, 18, 24),	(15, 23, 13, 11, 18),	(0, 14, 4, 8, 11),	(0, 8, 21, 17, 1),	
	(4, 16, 23, 2, ∞_0),	(15, 21, 12, 29, ∞_0),	(0, 2, 1, 13, ∞_0),	(0, 5, 10, 29, ∞_3),	+2 (mod 30).
$2^{15}6^1$:	(0, 16, 18, 14, 7),	(0, 25, 7, 17, 9),	(0, 18, 10, 11, ∞_0),	(2, 23, 27, 3, ∞_0),	
	(1, 20, 25, 28, ∞_0),	(17, 3, 4, 21, ∞_3),	(0, 13, 24, 4, ∞_3),	(2, 26, 29, 1, ∞_3),	+2 (mod 30).
$2^{17}4^1$:	(0, 4, 26, 16, 23),	(0, 13, 5, 33, 15),	(0, 31, 30, 25, ∞_0),	(0, 20, 11, 13, ∞_2),	(mod 34).
$2^{38}8^1$:	(0, 32, 21, 44, 2),	(0, 46, 20, 1, 6),	(0, 10, 11, 4, 8),	(0, 28, 68, 10, 41),	
	(0, 3, 42, 51, 64),	(0, 21, 35, 50, ∞_0),	(0, 25, 52, 9, ∞_2),	(0, 54, 7, 59, ∞_4),	
	(0, 16, 33, 53, ∞_6),				(mod 76).
4^86^1:	(0, 6, 26, 4, 11),	(0, 1, 13, 11, 15),	(0, 2, 29, 23, ∞_0),	(0, 3, 18, 15, ∞_1),	
	(0, 25, 12, 31, ∞_2),	(0, 18, 9, 19, ∞_3),	(0, 31, 22, 27, ∞_4),	(0, 4, 25, 7, ∞_5),	+2 (mod 32).
$4^{12}8^1$:	(0, 7, 41, 11, 13),	(0, 4, 10, 18, 39),	(0, 1, 16, 47, ∞_0),	(0, 10, 5, 27, ∞_2),	
	(0, 11, 8, 33, ∞_4),	(0, 16, 45, 25, ∞_6),			(mod 48).
$4^{13}6^1$:	(0, 32, 5, 46, 4),	(0, 34, 40, 37, 30),	(0, 28, 30, 7, 12),	(0, 19, 20, 11, ∞_0),	
	(0, 8, 43, 27, ∞_2),	(0, 21, 35, 50, ∞_4),			(mod 52).
$4^{20}2^1$:	(0, 32, 9, 42, 1),	(0, 34, 16, 21, 12),	(0, 6, 75, 62, 52),	(0, 76, 12, 47, 49),	
	(0, 59, 66, 11, 14),	(0, 44, 36, 19, 38),	(0, 24, 50, 7, 65),	(0, 27, 77, 26, ∞_0),	(mod 80).
$4^{23}6^1$:	(0, 36, 9, 44, 2),	(0, 54, 24, 41, 22),	(0, 6, 31, 16, 12),	(0, 16, 88, 70, 33),	
	(0, 3, 50, 43, 32),	(0, 10, 36, 15, 24),	(0, 51, 64, 21, 84),	(0, 40, 45, 1, ∞_0),	
	(0, 1, 35, 66, ∞_2),	(0, 64, 11, 89, ∞_4),			(mod 92).

$4^{28}6^1$:	(0, 88, 9, 92, 2),	(0, 54, 64, 41, 14),	(0, 6, 45, 8, 12),	(0, 8, 24, 42, 25),	
	(0, 3, 14, 63, 72),	(0, 50, 80, 15, 68),	(0, 55, 76, 25, 92),	(0, 36, 29, 93, 80),	
	(0, 26, 27, 65, 5),	$(0, 42, 23, 101, \infty_0)$,	$(0, 41, 107, 10, \infty_2)$,	$(0, 69, 38, 3, \infty_4)$,	(mod 112).
$4^{33}6^1$:	(0, 88, 29, 48, 2),	(0, 54, 64, 53, 50),	(0, 90, 49, 12, 80),	(0, 8, 20, 2, 17),	
	(0, 7, 30, 35, 48),	(0, 6, 32, 67, 112),	(0, 47, 116, 89, 28),	(0, 16, 17, 21, 72),	
	(0, 94, 72, 36, 57),	(0, 100, 23, 62, 92),	(0, 65, 7, 110, 31),	$(0, 25, 39, 82, \infty_0)$,	
	$(0, 98, 107, 51, \infty_2)$,	$(0, 70, 119, 11, \infty_4)$,			(mod 132).
$4^{38}6^1$:	(0, 88, 72, 43, 41),	(0, 33, 54, 23, 98),	(0, 132, 49, 48, 66),	(0, 58, 64, 53, 46),	
	(0, 82, 57, 4, 32),	(0, 8, 32, 2, 17),	(0, 3, 22, 27, 36),	(0, 10, 36, 91, 56),	
	(0, 63, 136, 89, 44),	(0, 40, 13, 117, 80),	(0, 130, 96, 100, 57),	(0, 92, 7, 58, 100),	
	(0, 113, 51, 110, 23),	$(0, 61, 11, 82, \infty_0)$,	$(0, 68, 55, 69, \infty_2)$,	$(0, 74, 123, 111, \infty_4)$,	(mod 152).
6^6:	(15, 3, 21, 9, 27),	(0, 18, 26, 29, 27),	$(0, 4, 2, 16, \infty_0)$,	$(2, 23, 15, 21, \infty_0)$,	
	$(1, 24, 13, 17, \infty_0)$,	$(0, 29, 6, 7, \infty_3)$,	$(0, 17, 8, 21, \infty_4)$,	$(0, 6, 17, 3, \infty_5)$,	+2 (mod 30).
$6^5 8^1$:	$(0, 6, 14, 2, \infty_0)$,	$(1, 7, 5, 23, \infty_0)$,	$(4, 3, 16, 27, \infty_0)$,	$(0, 14, 7, 3, \infty_3)$,	
	$(0, 2, 9, 1, \infty_4)$,	$(0, 3, 24, 21, \infty_5)$,	$(0, 26, 27, 13, \infty_6)$,	$(0, 13, 22, 11, \infty_7)$,	+2 (mod 30). □

Some of these will be used to obtain certain packing and covering designs (SPPDs and SPCDs) later in Section 7.

Lemma 3.7. *Suppose G is an abelian group of order gu, and there exists a set of base blocks (none of which is short) that generate a $(5, 1)$-GDD of type g^u when developed over G. Then there exists a super-simple HSPS of type g^u.*

Proof. Replace each base block $(a_0, a_1, a_2, a_3, a_4)$ by two base pentagons $(a_0, a_1, a_2, a_3, a_4)$ and $(-a_0, -a_3, -a_1, -a_4, -a_2)$, and develop these over G. In Gronau et al. [10, Theorem 2.2] it was noted that this gives a super-simple $(5, 2)$-GDD; with the pentagons ordered as given, any two points in different holes appear in one pentagon with distance 1 and one with distance 2, thus these pentagons also generate an HSPS(g^u). □

The HSPS of type 4^{16} in Lemma 3.3 is one example of an HSPS obtained by Lemma 3.7; the first 3 pentagons there generate a $(5, 1)$-GDD(4^{16}). Sometimes, but not always, this method can also produce a super-simple HSPS if the original $(5, 1)$-GDD was obtained over a group G plus one or more infinite points; see for example, the HSPS(4^{20}) in Lemma 3.3.

Lemma 3.8. *There exists a super-simple* SPS(v) *for* $v = 11, 21, 31, 41$.

Proof. For $v = 11$, we develop the following pentagon (mod 11) : (1, 3, 9, 5, 4). For $v = 31$, we develop the following 3 pentagons (mod 31) : (1, 2, 4, 8, 16), (5, 10, 20, 9, 18), (25, 19, 7, 14, 28). For $v = 21, 41$, we can apply Lemma 3.7, using a cyclic $(5, 1)$-GDD of type 1^v with base blocks (0, 1, 4, 14, 16) (for $v = 21$), or (1, 37, 16, 18, 10), (2, 33, 32, 36, 20) (for $v = 41$). □

Lemma 3.9. *There exists a super-simple* SPS(v) *for* $v = 35, 55, 135, 145$.

Proof. For $v = 35, 55$, let $X = Z_{v-1} \cup \{\infty\}$, and develop the pentagons below +2 (mod $(v - 1)$). For $v = 135, 145$, let $X = Z_{114} \cup \{\infty_0, \infty_1, \ldots, \infty_{t-1}\}$ for $t = 21$ or 31, and develop the pentagons below +2 (mod 114). Then form a super-simple SPS(t) on the infinite points. Also, for $v = 145$ and $z = 0, 3, 6, 9$, replace ∞_z by ∞_{z+x} $(x = 1, 2)$ whenever adding any value $\equiv 2x \pmod 6$ to a pentagon.

$v = 35$:	(0, 4, 10, 18, 7),	(0, 22, 25, 7, 33),	(0, 17, 26, 27, 5),	(0, 9, 28, 15, 11),
	(0, 20, 30, 32, 29),	(0, 27, 29, 23, 13),	$(0, 16, 1, 21, \infty)$.	
$v = 55$:	(0, 28, 8, 16, 7),	(0, 4, 27, 17, 53),	(0, 27, 24, 37, 31),	(0, 15, 18, 51, 47),
	(0, 48, 34, 9, 39),	(0, 30, 40, 23, 11),	(0, 17, 52, 50, 1),	(0, 32, 44, 26, 21),
	(0, 38, 19, 39, 25),	(27, 1, 3, 35, 43),	$(0, 9, 22, 11, \infty)$.	

$v = 135$:	$(3, 5, 17, 33, 51)$,	$(3, 33, 55, 59, 97)$,	$(0, 3, 71, 35, 11)$,	$(0, 85, 91, 101, 67)$,
	$(0, 4, 96, 28, 60)$,	$(0, 70, 84, 8, 10)$,	$(0, 64, 16, 22, 55)$,	$(0, 12, 20, 45, 77)$,
	$(0, 39, 4, 44, 45)$,	$(0, 34, 10, 31, 59)$,	$(0, 65, 14, 85, \infty_0)$,	$(0, 56, 15, 23, \infty_1)$,
	$(0, 9, 76, 19, \infty_2)$,	$(0, 18, 87, 27, \infty_3)$,	$(0, 31, 48, 67, \infty_4)$,	$(0, 36, 29, 43, \infty_5)$,
	$(0, 105, 88, 83, \infty_6)$,	$(0, 98, 11, 37, \infty_7)$,	$(0, 101, 112, 61, \infty_8)$,	$(0, 26, 79, 5, \infty_9)$,
	$(0, 15, 34, 3, \infty_{10})$,	$(0, 79, 102, 75, \infty_{11})$,	$(0, 84, 69, 17, \infty_{12})$,	$(0, 13, 46, 89, \infty_{13})$,
	$(0, 28, 77, 13, \infty_{14})$,	$(0, 72, 113, 55, \infty_{15})$,	$(0, 37, 32, 107, \infty_{16})$,	$(0, 61, 64, 63, \infty_{17})$,
	$(0, 52, 59, 103, \infty_{18})$,	$(0, 94, 73, 1, \infty_{19})$,	$(0, 29, 6, 95, \infty_{20})$.	
$v = 145$:	$(4, 6, 46, 14, \infty_0)$,	$(3, 11, 45, 41, \infty_0)$,	$(0, 68, 25, 61, \infty_0)$,	$(2, 8, 22, 40, \infty_3)$,
	$(5, 81, 13, 37, \infty_3)$,	$(0, 59, 66, 3, \infty_3)$,	$(4, 8, 30, 20, \infty_6)$,	$(3, 29, 9, 25, \infty_6)$,
	$(0, 1, 28, 35, \infty_6)$,	$(2, 26, 42, 4, \infty_9)$,	$(1, 51, 91, 63, \infty_9)$,	$(0, 52, 77, 59, \infty_9)$,
	$(0, 11, 73, 1, 13)$,	$(0, 58, 4, 89, 91)$,	$(0, 26, 14, 55, 65)$,	$(0, 20, 19, 44, 35)$,
	$(0, 39, 34, 53, \infty_{12})$,	$(0, 106, 85, 37, \infty_{13})$,	$(0, 17, 54, 57, \infty_{14})$,	$(0, 28, 11, 41, \infty_{15})$,
	$(0, 31, 84, 81, \infty_{16})$,	$(0, 30, 83, 97, \infty_{17})$,	$(0, 75, 52, 101, \infty_{18})$,	$(0, 44, 113, 91, \infty_{19})$,
	$(0, 83, 36, 17, \infty_{20})$,	$(0, 80, 9, 69, \infty_{21})$,	$(0, 63, 18, 99, \infty_{22})$,	$(0, 109, 6, 63, \infty_{23})$,
	$(0, 48, 75, 43, \infty_{24})$,	$(0, 29, 64, 49, \infty_{25})$,	$(0, 78, 111, 105, \infty_{26})$,	$(0, 67, 58, 95, \infty_{27})$,
	$(0, 64, 79, 21, \infty_{28})$,	$(0, 72, 93, 23, \infty_{29})$,	$(0, 101, 46, 5, \infty_{30})$.	

□

Lemma 3.10. *There exists a super-simple* SPS(v) *for* $v = 65, 85, 95, 115$.

Proof. Let $X = Z_{(v-11)} \cup \{\infty_0, \infty_1, \ldots, \infty_{10}\}$. Most base pentagons below should be developed $+2 \pmod{(v-11)}$; the exceptions are the first two pentagons for $v = 95, 115$, to which only the values $0, 2, 4, \ldots, (v-15)/2$ should be added. For $v = 95$, $z = 0$, and $v = 65$, $z = 0, 3$, replace ∞_z by ∞_{z+x} $(x = 1, 2)$ when adding any value $\equiv 2x \pmod 6$ to a base pentagon. For $v = 115$, replace ∞_0 by ∞_1 when adding any value $\equiv 2 \pmod 4$ to a base pentagon. Finally form a super-simple SPS(11) on the infinite points.

$v = 65$:	$(0, 48, 4, 32, 51)$,	$(0, 24, 10, 30, 9)$,	$(0, 31, 20, 35, 41)$,	$(27, 29, 7, 41, 13)$,
	$(0, 32, 15, 14, \infty_0)$,	$(1, 52, 47, 3, \infty_0)$,	$(28, 1, 30, 29, \infty_0)$,	$(0, 12, 8, 31, \infty_3)$,
	$(28, 35, 19, 11, \infty_3)$,	$(2, 4, 9, 27, \infty_3)$,	$(0, 8, 29, 41, \infty_6)$,	$(0, 38, 51, 27, \infty_7)$,
	$(0, 43, 26, 11, \infty_8)$,	$(0, 35, 42, 17, \infty_9)$,	$(0, 18, 9, 13, \infty_{10})$.	
$v = 85$:	$(0, 6, 52, 36, 12)$,	$(0, 38, 58, 24, 57)$,	$(0, 18, 10, 59, 61)$,	$(0, 1, 28, 53, 10)$,
	$(37, 33, 17, 27, 63)$,	$(0, 69, 29, 7, 21)$,	$(0, 52, 4, 11, 17)$,	$(37, 7, 53, 41, 56)$,
	$(0, 60, 27, 69, \infty_0)$,	$(37, 6, 19, 48, \infty_1)$,	$(0, 37, 2, 1, \infty_2)$,	$(0, 39, 62, 15, \infty_3)$,
	$(0, 4, 57, 49, \infty_4)$,	$(0, 5, 8, 37, \infty_5)$,	$(0, 2, 5, 23, \infty_6)$,	$(0, 32, 47, 67, \infty_7)$,
	$(0, 19, 26, 17, \infty_8)$,	$(0, 63, 54, 3, \infty_9)$,	$(0, 44, 55, 31, \infty_{10})$.	
$v = 95$:	$(3, 2, 44, 45, \infty_0)$,	$(0, 5, 47, 42, \infty_0)$,	$(64, 33, 25, 8, \infty_0)$,	$(13, 60, 64, 77, \infty_0)$,
	$(0, 32, 46, 8, 58)$,	$(3, 5, 51, 27, 20)$,	$(0, 2, 50, 32, 43)$,	$(0, 55, 37, 21, 47)$,
	$(0, 40, 7, 11, 59)$,	$(0, 15, 35, 16, 61)$,	$(3, 53, 21, 83, 56)$,	$(0, 8, 64, 70, 9)$,
	$(0, 30, 18, 78, 33)$,	$(0, 63, 51, 79, 65)$,	$(0, 81, 40, 75, \infty_3)$,	$(0, 68, 63, 57, \infty_4)$,
	$(0, 20, 5, 59, \infty_5)$,	$(0, 77, 74, 65, \infty_6)$,	$(0, 62, 61, 71, \infty_7)$,	$(0, 25, 4, 53, \infty_8)$,
	$(0, 71, 82, 27, \infty_9)$,	$(0, 10, 67, 23, \infty_{10})$.		
$v = 115$:	$(0, 5, 57, 52, \infty_0)$,	$(27, 2, 54, 79, \infty_0)$,	$(2, 47, 4, 41, \infty_0)$,	$(0, 4, 50, 12, 82)$,
	$(3, 5, 47, 51, 44)$,	$(0, 2, 42, 24, 47)$,	$(0, 3, 25, 17, 67)$,	$(0, 36, 9, 23, 87)$,
	$(0, 11, 27, 14, 15)$,	$(3, 9, 21, 39, 52)$,	$(0, 32, 88, 38, 73)$,	$(0, 14, 34, 44, 9)$,
	$(0, 49, 29, 87, 51)$,	$(0, 59, 28, 49, 96)$,	$(0, 6, 33, 59, 103)$,	$(0, 71, 37, 47, 17)$,
	$(0, 88, 58, 84, 65)$,	$(0, 79, 86, 63, \infty_2)$,	$(0, 92, 21, 93, \infty_3)$,	$(0, 19, 40, 11, \infty_4)$,
	$(0, 62, 103, 75, \infty_5)$,	$(0, 28, 19, 43, \infty_6)$,	$(0, 53, 56, 85, \infty_7)$,	$(0, 99, 10, 71, \infty_8)$,
	$(0, 80, 69, 31, \infty_9)$,	$(0, 44, 83, 35, \infty_{10})$.		

□

Lemma 3.11. *There exists a super-simple* SPS(v) *for* $v = 45, 75, 125, 165$.

Proof. For $v = 45$, let $X = Z_{44} \cup \{\infty\}$, and develop the following base pentagons $+4 \pmod{44}$:

(0, 1, 12, 16, 18), (0, 16, 33, 7, 43), (0, 11, 24, 22, 35), (0, 12, 36, 26, 15),
(0, 36, 5, 38, 37), (0, 22, 40, 34, 27), (0, 41, 9, 25, 3), (0, 29, 21, 42, 6),
(0, 21, 19, 39, 23), (0, 5, 43, 13, 30), (0, 19, 37, 41, 7), (0, 10, 17, 30, 14),
(33, 30, 31, 22, 3), (33, 9, 34, 29, 35), (33, 42, 38, 26, 23), (22, 2, 31, 43, 39),
$(0, 25, 10, 15, \infty)$, $(33, 24, 19, 42, \infty)$.

For $v = 75, 125$, let $X = Z_5 \times Z_{v/5}$, and develop the following pentagons (mod $(5, v/5)$). In both cases, the first two pentagons given are short and generate only $v/5$ pentagons each.

$v = 75$:
((0, 0), (0, 3), (0, 6), (0, 9), (0, 12)), ((0, 0), (1, 0), (2, 0), (3, 0), (4, 0)), ((0, 0), (0, 14), (3, 12), (2, 11), (4, 4)),
((0, 0), (0, 8), (0, 10), (2, 1), (3, 6)), ((0, 0), (1, 9), (1, 13), (4, 0), (2, 12)), ((0, 0), (1, 12), (3, 1), (3, 10), (4, 2)),
((0, 0), (2, 0), (3, 8), (1, 9), (3, 4)), ((0, 0), (4, 1), (1, 2), (0, 13), (3, 8)), ((0, 0), (1, 2), (0, 14), (1, 5), (1, 10)).

$v = 125$:
((0, 0), (0, 10), (0, 20), (0, 5), (0, 15)), ((0, 0), (2, 0), (4, 0), (1, 0), (3, 0)), ((0, 0), (1, 15), (4, 6), (1, 17), (2, 6)),
((0, 0), (4, 24), (1, 2), (2, 14), (4, 15)), ((0, 0), (4, 2), (4, 4), (1, 6), (3, 5)), ((0, 0), (2, 22), (4, 15), (2, 11), (3, 8)),
((0, 0), (2, 13), (2, 8), (0, 17), (0, 16)), ((0, 0), (2, 21), (1, 1), (2, 18), (1, 7)), ((0, 0), (2, 19), (2, 5), (4, 17), (3, 17)),
((0, 0), (4, 23), (3, 18), (3, 5), (2, 14)), ((0, 0), (1, 4), (3, 9), (0, 24), (1, 8)), ((0, 0), (4, 12), (3, 19), (3, 22), (0, 7)),
((0, 0), (0, 19), (4, 16), (0, 12), (4, 6)), ((0, 0), (4, 1), (4, 22), (2, 24), (2, 7)).

For $v = 165$, let $X = Z_{154} \cup \{\infty_0, \infty_1, \ldots, \infty_{10}\}$. Develop the last pentagon below $(+2 \bmod 154)$, and all others $(\bmod 154)$. Also replace ∞_0 by ∞_i when adding any value $\equiv i \pmod{11}$ to a pentagon. This gives a super-simple HSPS of type 11^{15}; filling in each size 11 hole with a super-simple SPS(11) then gives a super-simple SPS(165).

(0, 103, 43, 86, 81), (0, 147, 39, 87, 9), (0, 113, 130, 8, 47), (0, 33, 99, 2, 134),
(0, 141, 48, 32, 8), (0, 151, 52, 148, 34), (0, 12, 92, 73, 38), (0, 91, 135, 82, 71),
(0, 29, 97, 103, 18), (0, 118, 36, 95, 105), (0, 37, 12, 10, 133), $(0, 27, 4, 34, \infty_0)$,
$(2, 106, 91, 65, \infty_0)$, $(9, 118, 53, 117, \infty_0)$, $(8, 12, 112, 37, \infty_0)$, $(3, 55, 142, 50, \infty_0)$,
$(5, 6, 83, 82, \infty_0)$. □

4. Recursive constructions

In the recursive constructions of group divisible designs (GDDs) and pairwise balanced designs (PBDs), the "weighting" technique and Wilson's fundamental construction (see [14]) are quite often used, where we start with a "master" GDD and small input designs to obtain a new GDD. Similar techniques will be applied in our constructions of super-simple HSPSs, where we either start with a super-simple HSPS and use transversal designs (TDs) as input designs or start with a GDD and use some super-simple HSPSs as input designs. More specifically, we shall make use of the following two constructions, which are just slight modifications of those found in [1]. The second is a more general form of Construction 3.3 in [1]. For more background knowledge about GDDs and PBDs the readers are referred to [6].

Construction 4.1. *Suppose that both a super-simple HSPS of type* $\{h_1, h_2, \ldots, h_n\}$ *and a TD*$(5, m)$ *exist. Then there exists a super-simple HSPS of type* $\{mh_1, mh_2, \ldots, mh_n\}$.

Construction 4.2. *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a GDD with groups* $G_1, G_2, \ldots, G_n$. *Suppose there exists a function* $w : (X \to \mathbf{Z}^+ \cup \{0\}$ *(a weighting function), which has the property that for every block* $B = \{x_1, \ldots, x_k\} \in \mathcal{B}$ *there exists a super-simple HSPS of type* $(w(x_1), \ldots, w(x_k))$. *Then there exists an super-simple HSPS of type* $\{\sum_{x \in G_1} w(x), \ldots, \sum_{x \in G_n} w(x)\}$.

In particular, when the input GDD is a PBD, and all weights are uniform, this gives us the following result:

Construction 4.3. *If a (v, K)-PBD exists and there exists a super-simple HSPS of type g^k for all $k \in K$, then there exists a super-simple HSPS of type g^v.*

In the construction of GDDs or PBDs, the technique of "filling in holes" plays an important role. The technique for super-simple SPSs and HSPSs is described as follows.

Construction 4.4 (*Filling in Holes*). *Suppose there exists a super-simple HSPS of type $\{s_i : 1 \leqslant i \leqslant n\}$. Let $a \geqslant 0$ be an integer. For $1 \leqslant i \leqslant n-1$, if there exists a super-simple ISPS$(a+s_i, a)$, then there exists a super-simple ISPS$(a+s, a+s_n)$, where $s = \sum_{i=1}^{n} s_i$. Moreover, if there exists a super-simple SPS$(a+s_n)$, then there exists a super-simple SPS$(a+s)$.*

5. Super-simple SPS(v) for $v \equiv 1 \pmod{10}$

We start by giving three known results for TDs which will frequently be used:

Lemma 5.1 (*Abel et al. [3, p.163]*). *There exists a TD$(5, m)$ for every positive integer $m \neq 2, 3, 6, 10$.*

Lemma 5.2 (*Abel et al. [3, p.163]*). *There exists a TD$(6, m)$ for every positive integer $m \geqslant 5$ and $m \neq 6, 10, 14, 18, 22$.*

Lemma 5.3. *If a TD(k, m) exists, then (1) a $\{k, k-1, m\}$-GDD of type $(k-1)^m u^1$ exists for $0 \leqslant u \leqslant m-1$, and (2) a $\{k, k-1, m+1\}$-GDD of type $(k-1)^m u^1$ exists for $1 \leqslant u \leqslant m$.*

Proof. Truncate one group of the TD to size u (in case (1)) or $u-1$ (in case (2)) and (in case (2) only) add 1 extra point to the groups. Using a deleted point to redefine groups now gives the required design. □

Lemma 5.4. *If $n \geqslant 5$ and $n \notin \{14, 23\}$, then a super-simple HSPS of type 10^n exists. If $n \geqslant 5$, then a super-simple ISPS$(10n+1, 11)$ and a super-simple SPS$(10n+1)$ both exist.*

Proof. We start by constructing super-simple HSPSs of type 10^n. When these exist, a super-simple SPS$(10n+1)$ can be obtained by filling in all groups with an extra point, using a super-simple SPS(11). A super-simple ISPS$(10n+1, 11)$ can be obtained similarly, by filling in all groups except one.

For types 10^5, 10^7 and 10^8, see Lemma 3.4. For $n = 6, 10, 11, 15, 16, 20, 21$ we can inflate a super-simple HSPS(2^n) (from Lemma 3.2) by 5. For $n = 9, 13, 17$, we can apply Lemma 3.7; the required cyclic (5, 1)-GDDs of type 10^n with no short blocks or infinite points can be found in [15]. For $n = 12$, deleting 1 point from a TD(11, 11) gives an 11-GDD(10^{12}); inflate this by 1, using HSPS(1^{11}). For $n = 18, 19, 22, 24, 25, 27, 28, 30, 31, 34$, we delete 1 point from a $(5n+1, 6, 1)$-BIBD to obtain a 6-GDD(5^n), and inflate it, using a super-simple HSPS(2^6). For $n = 32$, start with a TD(6, 26), delete one block, and on each group plus 5 extra points form a (31, 6, 1)-BIBD missing 1 block (with the missing block containing that group's point from the deleted block of the TD and the 5 extra points). Finally form a block of size 11 on the 6 points in the deleted block of TD(6, 26) and the 5 extra points. This gives a $(161, \{6, 11^*\})$-PBD. Deleting a point not in the size 11 block gives a $\{6, 11\}$-GDD(5^{32}). Now inflate this, using super-simple HSPSs of types 2^6 and 2^{11}.

For $n = 29, 33$, inflate super-simple HSPSs of types 10^7 and 10^8 with TD(5, 4), giving super-simple HSPSs of types 40^7 and 40^8. Now add 10 points and form a super-simple HSPS of type 10^5 on each hole plus the 10 extra points. This gives super-simple HSPSs of types 10^{29} and 10^{33}. For $n = 26$ or $n > 34$, we can apply Construction 4.2, starting with a $(n, \{5, 6, 7, 8, 9\})$-PBD (which exists, [4]), and inflating it using super-simple HSPSs of type 10^t ($t = 5, 6, 7, 8, 9$).

For $n = 23$, inflate a super-simple HSPS(1^{21}) by 11. For $n = 14$, inflating the super-simple HSPS$(2^{12}4^1)$ in Lemma 3.6 by 5 gives a super-simple HSPS$(10^{12}20^1)$. Now, using an extra point, fill in all holes for a super-simple SPS(141), and all holes except one of size 10 for a super-simple ISPS(141, 11). □

Combining Lemmas 3.8 and 5.4, we have the following:

Theorem 5.5. *If $v \equiv 1 \pmod{10}$, then a super-simple SPS(v) exists.*

6. Super-simple SPS(v) for $v \equiv 5 \pmod{10}$

We have just shown that for $n \geqslant 1$, there exists a super-simple SPS$(10n+1)$ and for $n \geqslant 5$, there exists a super-simple ISPS$(10n+1, 11)$. Also, if $n \geqslant 5$ and $n \neq 14$ or 23, then a super-simple HSPS(10^n) exists. For $v \equiv 5 \pmod{10}$, $35 \leqslant v \leqslant 165$ and $v \notin \{105, 155\}$, super-simple SPS(v)s were obtained directly in Lemmas 3.9–3.11.

These results will be of major significance when proving existence of super-simple SPS(v) with $v \equiv 5 \pmod{10}$. We start by concentrating on the smaller values, or more specifically, the cases $v \equiv i \pmod{100}$, and $v < p(i)$ where the values i and $p(i)$ are given in the table below. Larger values of v will be handled later in Lemma 6.13.

i	$p(i)$	i	$p(i)$
15	715	5	705
35	335	25	825
55	355	45	345
75	475	65	465
95	595	85	585

We now give a few general constructions:

Lemma 6.1. *If there exists a super-simple $TD_2(5, q)$ which is the union of two* TD$(5, q)$*s, then there exists a super-simple HSPS of type q^5.*

Proof. Let $I_m = \{0, 1, 2, \ldots, m-1\}$, and label the points of the TD as (x, y) with $x \in I(5)$ and $y \in I(q)$. The required HSPS is obtained by leaving each block $((0, a_0), (1, a_1), (2, a_2), (3, a_3), (4, a_4))$ in the first TD$(5, q)$ unaltered, and replacing each such block in the second TD$(5, q)$ by $((0, a_0), (2, a_2), (4, a_4), (1, a_1), (3, a_3))$. □

Lemma 6.2. *If q is an odd prime power, then there exists a super-simple $TD_2(q, q)$ which is the union of two TD(q, q)s.*

Proof. Let A and B be $q \times q$ matrices, over GF(q) with rows and columns indexed by the elements of GF(q). Let a and b be any two distinct non-zero elements of GF(q), and let $A(x, y) = a(x-y)^2$, $B(x, y) = b(x-y)^2$.

Then A and B are $(q, q, 1)$ difference matrices, and the $TD_2(q, q)$ generated by the difference matrix $E = [A|B]$ will be super-simple if for every pair of distinct columns c and d of E, and for every element z of GF(q), the equation $c(x) - d(x) = z$ has at most two solutions for x. (If $u_1, u_2, \ldots, u_k$ are the row indices of E and $t_1, t_2, \ldots, t_{2q}$ are the column indices of E, then this $TD_2(q, q)$ is over $\{u_1, u_2, \ldots, u_k\} \times GF(q)$, and is obtained by developing the second coordinates of the points in the $2q$ blocks $\{(u_1, E(u_1, t_i)), (u_2, E(u_2, t_i)), \ldots, (u_k, E(u_k, t_i))\}$ $(i = 1, 2, \ldots, 2q)$ over GF(q).) However, it is not hard to see that there are at most 2 solutions for any x, since for any such columns c, d, $c(x) - d(x)$ is a polynomial function of x of degree 1 or 2. □

Combining the preceding two lemmas, we have:

Lemma 6.3. *If $q \geqslant 5$ is an odd prime power, then there exists a super-simple HSPS of type q^5.*

Corollary 6.4. *If $v \in \{155, 205, 305, 405, 505, 605\}$, then there exists a super-simple HSPS of type $(v/5)^5$ and a super-simple SPS(v).*

Lemma 6.5. *There exists a super-simple SPS(v) for $v \in \{175, 225, 275, 325, 375, 425, 625\}$.*

Proof. Inflate the super-simple HSPS(5^5) in Lemma 3.1 with a TD$(5, m)$ for $m = 7, 9, 11, 13, 15, 17$ or 25, to obtain super-simple HSPSs of types 35^5, 45^5, 55^5, 65^5, 75^5, 85^5, 125^5 and fill in the holes. □

We now give two more general constructions, which are helpful in dealing with several other small values:

Construction 6.6. *Suppose a TD$(6, m)$ exists and u is even. Then*:

1. *If* $m \equiv 0$ *or* $4 \pmod 5$, $u \notin \{10, 14\}$, $8 \leqslant u \leqslant 10(m-1)+8$ *and a super-simple HSPS*(8^{m+1}) *exists*, *then a super-simple HSPS*$(40^m u^1)$ *also exists*;
2. *If* $m \equiv 0$ *or* $1 \pmod 5$, $u \notin \{2, 6\}$, $0 \leqslant u \leqslant 10(m-1)$, *and a super-simple HSPS*(8^m) *exists*, *then a super-simple HSPS*$(40^m u^1)$ *also exists*;
3. *If in either of the above cases a super-simple SPS*$(u+1)$ *exists*, *then a super-simple SPS*$(40m+u+1)$ *exists*.

Proof. Start with a TD$(6, m)$, and for $K = \{5, 6, m+1\}$ (in case (1)) or $\{5, 6, m\}$ (in case (2)) we apply Lemma 5.3 to obtain a K-GDD of type $5^m m^1$ (in case (1)) or $5^m (m-1)^1$ (in case (2)). Since we have super-simple HSPSs of types $8^5 t^1$ for $t = 0, 4, 8, 10$ (by Lemma 3.5), as well as one of type 8^{m+1} or 8^m, we may apply Construction 4.2, giving weight 8 to all points in the size 5 groups, and (in case 1 only) weight 8 to the 'extra' point in group of size m. Give other points in the group of size m or $m-1$ weight 0, 4, 8 or 10 so that the total weight of points in this group is u. This gives a super-simple HSPS$(40^m u^1)$. Finally, fill in the holes with one extra point. □

Lemma 6.7. *If* $v \in \{415, 485, 495, 515, 525, 725\}$, *then a super-simple SPS*(v) *exists*.

Proof. Apply Construction 6.6, using the parameters indicated. The required super-simple HSPSs of type 8^m ($m = 10, 11, 16$) can be obtained by Construction 4.1, inflating a super-simple HSPS of type 2^m from Lemma 3.2 with a TD$(5, 4)$.

v	m	K	HSPS obtained	v	m	K	HSPS obtained
415	9	{5, 6, 10}	$40^9 54^1$	515	11	{5, 6, 11}	$40^{11} 74^1$
485	11	{5, 6, 11}	$40^{11} 44^1$	525	11	{5, 6, 11}	$40^{11} 84^1$
495	11	{5, 6, 11}	$40^{11} 54^1$	725	16	{5, 6, 16}	$40^{16} 84^1$ □

Construction 6.8. *If a TD*$(6, m)$ *exists*, u *is even*, $0 \leqslant u \leqslant 10m$ *and* $u \notin \{2, 6\}$, *then a super-simple HSPS of type* $(8m)^5 u^1$ *exists. If further, super-simple SPSs of orders* $8m+1$ *and* $u+1$ *exist*, *then a super-simple SPS*$(40m+u+1)$ *also exists*.

Proof. Start with a TD$(6, m)$ and give weight 8 to all points in the first 5 groups. Let $u = 4x + 8y + 10z$ where x, y, z are non-negative integers with $x + y + z \leqslant m$, and in the last group, we can give weight 4 to x points, weight 8 to y points weight 10 to z points, and zero weight to the rest. Since super-simple HSPSs of types $8^5 u^1$ exist for $u = 0$, 4, 8, 10 by Lemma 3.5, this gives a super-simple HSPS of type $(8m)^5 u^1$. We now complete the construction by filling in the holes with 1 extra point, using super-simple SPSs of orders $8m+1$ and $u+1 = 4x+8y+10z+1$. □

Lemma 6.9. *If* $v \in \{235, 245, 365, 385, 395, 615\}$, *then a super-simple SPS*(v) *exists*.

Proof. Apply Construction 6.8, using the values of m and u given in the following table.

v	m	u	v	m	u
235	5	34	385	8	64
245	5	44	395	8	74
365	8	44	615	13	94 □

The next two lemmas provide a couple of miscellaneous constructions:

Lemma 6.10. *If* $v \in \{105, 185, 255, 265, 315\}$, *then a super-simple SPS*(v) *exists.*

Proof. For 185, start with a TD(6, 15) and adjoin a point x to the groups. Now delete some other point and use its blocks to redefine groups. This gives a {6, 16}-GDD of type $5^{15}15^1$. Now give weight 6 to x and weight 2 to all other points. Since super-simple HSPSs of types 2^6 and $2^{15}6^1$ exist by Lemmas 3.2 and 3.6, this gives a super-simple HSPS of type $10^{15}34^1$. Filling in the holes of this HSPS with an extra point now gives a super-simple SPS(185).

For 105, 265, 255 and 315, we inflate super-simple HSPSs of types 3^5, 4^6, 3^5 and 5^7 from Lemmas 3.3 and 3.1 with a TD(5, t) for $t = 7, 11, 17, 9$ to obtain super-simple HSPSs of types 21^5, 44^6, 51^5 and 45^7. We now fill in the groups with 1 or 0 extra points using super-simple SPS(g)s for $g = 21$, 45, 51 and 45. □

Lemma 6.11. *A super-simple SPS*(v) *exists for* $v \in \{195, 215, 285, 295\}$.

Proof. For 195, 215, 285 and 295, start with TD(6, 8), TD(6, 9), TD(6, 12) and TD(6, 13), respectively; add an infinite point y to the groups, then delete a different point so as to form a {6, 9}-GDD of type $5^8 8^1$, a {6, 10}-GDD of type $5^9 9^1$, a {6, 13}-GDD of type $5^{12}12^1$ and a {6, 14}-GDD of type $5^{13}13^1$. Give weight 4 to all points in the groups of size 5. For $v = 195$, in the 8-group, give weight 6 to y, and weight 4 to the other 7 points. For $v = 215$, in the 9-group, give weight 4 to y, weight 4 to 7 points, and zero weight to 1 point. For $v = 285$, in the 12-group, give weight 8 to y, weight 4 to 9 other points and zero weight to the rest. For $v = 295$, in the 13-group, we give weight 6 to y, weight 4 to 7 other points and zero weight to the rest. By using super-simple HSPSs of types 4^5, 4^6, $4^8 6^1$, 4^{10}, $4^{12}8^1$ and $4^{13}6^1$, we obtain super-simple HSPSs of types $20^8 34^1$, $20^9 32^1$, $20^{12}44^1$ and $20^{13}34^1$. Filling in the holes of these designs (with 3 extra points for $v = 215$, and 1 extra point in the other cases) we obtain super-simple SPSs for $v = 195$, 215, 285 and 295. For $v = 215$, the super-simple ISPS(23, 3) required for the fill in is given in Lemma 3.6. □

So far in this section, we have concentrated on super-simple SPSs with $v \equiv i \pmod{100}$, $v < p(i)$, (where for $i = 5, 15, \ldots, 95$, the values $p(i)$ were given at the start of this section, and are given again in Lemma 6.13). With the aid of the super-simple SPSs given so far, it is not hard to prove existence of super-simple SPSs with $v \equiv i \pmod{100}$, $v \geqslant p(i)$. The next lemma is a powerful tool for dealing with this result:

Lemma 6.12. *If* $0 \leqslant t \leqslant 5m$ *and* $m \geqslant 3$, *then a super-simple HSPS of type* $(20m)^5(4t)^1$ *also exists*. *Further*:

1. *If a super-simple SPS*$(4t + 1)$ *exists*, *then a super-simple SPS*$(100m + 4t + 1)$ *exists*;
2. *If a super-simple SPS*$(4t + 11)$ *exists*, *then a super-simple SPS*$(100m + 4t + 11)$ *exists*.

Proof. A TD(6, 5m) exists by Lemma 5.2 when $m \geqslant 3$; truncate one group to size t. Since super-simple HSPSs of types 4^5 and 4^6 exist (by Lemma 3.3), we may apply Construction 4.2, giving weight 4 to all points to obtain a super-simple HSPS of type $(20m)^5(4t)^1$. Now apply Construction 4.4, filling in the holes with 1 or 11 extra points. The required super-simple SPS($20m + 1$) and super-simple ISPS($20m + 11$, 11) both exist by Lemma 5.4. □

Lemma 6.13. *If* $v \equiv i \pmod{100}$, *and* $v \geqslant p(i)$, *where* i, $p(i)$ *are as given in the table below*, *then a super-simple SPS*(v) *exists.*

Proof. Apply Lemma 6.12, using the parameters indicated.

$i \pmod{100}$	(t, $4t + 1$ or $4t + 11$)	Range for m	$p(i)$
15	(26, 115)	$m \geqslant 6$	715
35	(6, 35)	$m \geqslant 3$	335
55	(11, 55)	$m \geqslant 3$	355
75	(16, 75)	$m \geqslant 4$	475
95	(21, 95)	$m \geqslant 5$	595

5	(26, 105)	$m \geqslant 6$	705
25	(31, 125)	$m \geqslant 7$	825
45	(11, 45)	$m \geqslant 3$	345
65	(16, 65)	$m \geqslant 4$	465
85	(21, 85)	$m \geqslant 5$	585 □

We have now proved the existence results for super-simple SPS(v) stated in Section 1. As a consequence, we have also completely solved the existence problem in Theorem 1.1 for super-simple $(v, 5, 2)$-BIBDs. For convenience, we restate these in the following theorems:

Theorem 6.14. *There exists a super-simple SPS*(v) *for all* $v \equiv 1, 5 \pmod{10}$, *except* $v = 5, 15$ *and possibly* $v = 25$.

Theorem 6.15. *There exists a super-simple* $(v, 5, 2)$*-BIBD if and only if* $v \equiv 1$ *or* $5 \pmod{10}$, *except for* $v \in \{5, 15\}$.

7. An application to Steiner Pentagon packings and coverings

Apart from the above mentioned results, there are some other constructions related to designs we have provided in this paper that are worth mentioning. In what follows, K_n denotes the complete undirected graph on n vertices.

A *Steiner pentagon covering* (SPC) of order n is a pair $(K_n, \mathscr{B})$, where $\mathscr{B}$ is a collection of pentagons from K_n such that any two vertices are joined by a path of length 1 in at least one pentagon of $\mathscr{B}$, and also by a path of length 2 in at least one pentagon of $\mathscr{B}$. It is well known that any SPS of order n gives a BIBD on n points with block size $k = 5$ and index $\lambda = 2$. Similarly, an SPC of order n may lead to a usual covering on n points with $k = 5$ and index $\lambda = 2$. It is known that such a covering contains at least

$$c(n) = \left\lceil \frac{n}{5} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil$$

blocks. If an SPC(n) contains the minimum number of $c(n)$ pentagons, we call it a *Steiner pentagon covering design* (SPCD), denoted by SPCD(n).

A *Steiner pentagon packing* (SPP) of order n is a pair $(K_n, \mathscr{B})$, where $\mathscr{B}$ is a collection of pentagons from K_n such that any two vertices are joined by a path of length 1 in at most one pentagon of $\mathscr{B}$, and also by a path of length 2 in at most one pentagon of $\mathscr{B}$. As mentioned earlier, any SPS of order n gives a BIBD on n points with block size $k = 5$ and index $\lambda = 2$. An SPP of order n may lead to the ordinary packing on n points with $k = 5$ and index $\lambda = 2$. It is known that such a packing contains at most

$$p(n) = \left\lfloor \frac{n}{5} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - \varepsilon$$

blocks, where $\varepsilon = 1$ for $n \equiv 7$ or $9 \pmod{10}$ and $\varepsilon = 0$ otherwise. If an SPP(n) contains the maximum number of $p(n)$ pentagons, we call it a *Steiner pentagon packing design* (SPPD), denoted by SPPD(n).

The following existence results for SPPDs and SPCDs were given in [2] and [5], and the reader is referred to these papers for more detailed information.

Theorem 7.1 (*Abel et al. [2]*). *An SPCD*(v) *exists for all* $v \geqslant 5$ *except for* $v = 9, 15$, *and possibly for* $v \equiv 3 \pmod{10}$ *or* $v \in \{14, 18, 22, 28, 38, 42, 74, 82\} \cup \{19, 27, 29, 37, 99, 119, 139, 159\}$.

Theorem 7.2 (*Bennett et al. [5], Abel et al. [2]*). *An SPPD*(v) *exists for all* $v \geqslant 5$ *except for* $v = 9, 13, 15$, *and possibly for* $v \in \{16, 18, 24, 34, 36, 84\} \cup \{17, 19, 27, 29, 33, 37, 69, 73, 77, 83, 89, 99, 109, 119, 129, 139, 149, 159, 169, 189\}$.

Some new SPPDs and SPCDs can be obtained from certain HSPSs given in Lemma 3.6. Thus an SPPD(189) can be obtained using the HSPSs of types 6^6, $6^5 8^1$. Start with a TD(6, 5), and give weight 6 to all points in the first 5 groups; in the 6th group give weight 8 to 4 points and weight 6 to the other point. Inflating using HSPSs of types 6^6 and $6^5 8^1$ gives an HSPS of type $30^5 38^1$. Filling in the holes with 1 extra point (using SPS(31) or SPPD(39)) gives an

SPPD(189). In Lemma 5.7 of [2], it is shown that an SPCD($6m+2$) exists whenever an HSPS of type 6^m exists; hence we now have an SPCD(38). We also note that by filling in the hole of size 6 in HSPS($2^{15}6^1$) with a Steiner pentagon packing design (SPPD) of order 6, one can obtain a new SPPD of order 36, while HSPSs of types $1^{30}3^1$, $2^{15}4^1$, $1^{62}7^1$, $1^{70}3^1$, $1^{82}7^1$, and $1^{102}7^1$ give SPPDs of orders 33, 34, 69, 73, 89 and 109, respectively. For 69, 89 and 109, the size 7 hole should be filled with an SPPD(7). In addition, HSPSs of types 4^n6^1, $n = 23, 28, 33, 38$ give SPPDs (and SPCDs) of orders 99, 119, 139, 159; here we use an extra point together with SPS(5) and SPPD(7) or SPCD(7) to fill in the holes. Filling in the size 4 holes of HSPS($4^{20}2^1$) with an extra point and an SPS(5) gives a SPPD(83). Finally, filling in the size 8 hole in HSPS($2^{38}8^1$) with an SPPD(8) gives an SPPD(84).

We can now give the following updates to Theorems 7.1 and 7.2:

Theorem 7.3. *An SPCD*(v) *exists for all* $v \geqslant 5$ *except for* $v = 9, 15$, *and possibly for* $v \equiv 3 \pmod{10}$ *or* $v \in \{14, 18, 22, 28, 42, 74, 82\} \cup \{19, 27, 29, 37\}$.

Theorem 7.4. *An SPPD*(v) *exists for all* $v \geqslant 5$ *except for* $v = 9, 13, 15$, *and possibly for* $v \in \{16, 18, 24\} \cup \{17, 19, 27, 29, 37, 77, 129, 149, 169\}$.

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